

ON THE COMPLETE INTEGRABILITY OF THE PERIODIC QUANTUM TODA LATTICE

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ABSTRACT. We prove that the periodic quantum Toda lattice corresponding to any extended Dynkin diagram is completely integrable. This has been conjectured and proved in all classical cases and E_6 by Goodman and Wallach at the beginning of the 1980's. As a direct application, in the context of quantum cohomology of affine flag manifolds, results that were known to hold only for some particular Lie types can now be extended to all types.

1. STATEMENT OF THE MAIN RESULT

We first recall the notion of “ $ax + b$ ” Lie algebra, as defined by Goodman and Wallach in [6]. (In fact, the definition in that paper is more general: we only present here a special case which is relevant for our goal, which is to establish the complete integrability of the periodic quantum Toda lattice.) Let \mathfrak{a} and \mathfrak{u} be two finite dimensional real vector spaces and endow $\mathfrak{b} := \mathfrak{a} \oplus \mathfrak{u}$ with a Lie bracket $[\cdot, \cdot]$ and an inner product $\langle \cdot, \cdot \rangle$ such that:

1. \mathfrak{a} is orthogonal to \mathfrak{u} ;
2. \mathfrak{a} is a Lie subalgebra of \mathfrak{b} , \mathfrak{u} is an ideal of \mathfrak{b} , and both \mathfrak{a} and \mathfrak{u} are commutative Lie algebras;
3. for each $H \in \mathfrak{a}$, $\text{ad}(H)|_{\mathfrak{u}}$ is a self-adjoint endomorphism of \mathfrak{u} ;
4. one has the eigenspace decomposition $\mathfrak{u} = \bigoplus_{\alpha \in \Psi} \mathfrak{u}_{\alpha}$, where

$$\mathfrak{u}_{\alpha} := \{X \in \mathfrak{u} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\},$$

for some finite subset $\Psi \subset \mathfrak{a}^*$, such that \mathfrak{u}_{α} are all one-dimensional vector spaces;

5. there exists an irreducible root system $\Pi \subset \mathfrak{a}^*$, a system of simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\} \subset \Pi$, and a dominant root $\beta \in \Pi$ such that $\Psi = \Delta \cup \{-\beta\}$.

Recall that, by definition, a root β is *dominant* if $\langle \beta, \alpha_i \rangle \geq 0$ for all $1 \leq i \leq r$. If the root system Π is simply laced, there exists a unique dominant root, which is the highest root. If Π is of one of the types B_r, C_r, F_4 , or G_2 , then there are two dominant roots: one is long (i.e., the highest root) and the other is short. We refer to [6, Fig. 5.1] for the precise description. The Lie algebra \mathfrak{b} is essentially determined by the root system Π , the simple roots Δ , and the dominant root β , see [6, p. 362]. Let $U(\mathfrak{b})$ be the universal enveloping algebra of \mathfrak{b} , which

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comes equipped with a canonical grading. If Z_1, \dots, Z_{2r+1} is a basis of \mathfrak{b} and $(c^{ij})_{1 \leq i, j \leq 2r+1}$ the matrix inverse to $(\langle Z_i, Z_j \rangle)_{1 \leq i, j \leq 2r+1}$ then the element

$$\Omega := \sum_{i, j=1}^{2r+1} c^{ij} Z_i Z_j$$

of $U_2(\mathfrak{b})$ is independent of the choice of the basis and is called the *Laplacian* of \mathfrak{b} , see, e.g., [1, Proposition 1.3.1].

Denote $\alpha_0 := -\beta$ and consider an orthonormal basis X_0, X_1, \dots, X_r of \mathfrak{u} such that $X_i \in \mathfrak{u}_{\alpha_i}$, for all $0 \leq i \leq r$. Also consider $H_1, \dots, H_r \in \mathfrak{a}$ such that

$$\alpha_i(H_j) = \delta_{ij}, \text{ for all } 1 \leq i, j \leq r.$$

One has

$$(1.1) \quad \Omega = \sum_{i, j=1}^r \langle \alpha_i, \alpha_j \rangle H_i H_j + \sum_{i=0}^r X_i^2,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on \mathfrak{a}^* induced by the canonical isomorphism $\mathfrak{a}^* \simeq \mathfrak{a}$. By the PBW theorem, $U(\mathfrak{b})$ has a basis given by monomials of the form $X_{i_1} \cdots X_{i_k} H_{j_1} \cdots H_{j_\ell}$ where $0 \leq i_1, \dots, i_k \leq r$ and $1 \leq j_1, \dots, j_\ell \leq r$. Let $U(\mathfrak{b})_{\text{ev}}$ be the linear subspace of $U(\mathfrak{b})$ spanned by the monomials of the form $X_{i_1}^2 \cdots X_{i_k}^2 H_{j_1} \cdots H_{j_\ell}$ where $0 \leq i_1, \dots, i_k \leq r$ and $1 \leq j_1, \dots, j_\ell \leq r$. The projection $\mathfrak{b} \rightarrow \mathfrak{a}$ is a Lie algebra homomorphism and induces an algebra homomorphism $\mu : U(\mathfrak{b}) \rightarrow U(\mathfrak{a}) = \text{Sym}(\mathfrak{a})$ called the *symbol map*. Let W be the Weyl group of Π , which acts on \mathfrak{a} , and consequently also on the symmetric algebra $\text{Sym}(\mathfrak{a})$. Choose fundamental homogeneous generators u_1, \dots, u_r of $\text{Sym}(\mathfrak{a})^W$; by convention take $u_1 := \sum_{i,j=1}^r \langle \alpha_i, \alpha_j \rangle H_i H_j$.

The goal of this note is to prove the following result, which was conjectured by Goodman and Wallach in [6]. (We say that $\Gamma \in U(\mathfrak{b})$ has *degree* n if $\Gamma \in U_n(\mathfrak{b}) \setminus U_{n-1}(\mathfrak{b})$, where $U_0(\mathfrak{b}) \subset U_1(\mathfrak{b}) \subset \dots$ is the canonical filtration of the universal enveloping algebra.)

Theorem 1.1. *For any $i \in \{1, \dots, r\}$ there exists a unique $\Omega_i \in U(\mathfrak{b})_{\text{ev}}$ such that:*

- $[\Omega_i, \Omega] = 0$;
- Ω_i has degree equal to $\deg u_i$;
- the symbol of Ω_i is u_i , the chosen generator of $\text{Sym}(\mathfrak{a})^W$.

Furthermore, for any $i, j \in \{1, \dots, r\}$ one has $[\Omega_i, \Omega_j] = 0$, where $\Omega_1 := \Omega$.

The theorem was proved in [6, 7] in the cases when Π is of type A_r, B_r, C_r, D_r , or E_6 (note that when Π is of type B_r or C_r , there are two situations to be considered: when β is a long root and when β is a short root). The results are nicely summarized in [7, Theorem 3.3]. The importance of Theorem 1.1 resides in that it automatically implies the complete integrability of the periodic quantum Toda lattice associated to the extended Dynkin diagram $\Delta \cup \{-\beta\}$, as explained in [6]. More recently, in the case when β is the highest root, the aforementioned integrability was established by Etingof in [3]. Thus the only remaining cases are when Π is

of type F_4 or G_2 and β is short. These are the cases that are addressed in the main body of this paper. For the sake of completeness and clarity we have considered necessary to see exactly how Etingof's approach can be adapted to prove Theorem 1.1 in the case when β is the highest root: this is done in Appendix A.

Remark 1.2. Theorem 1.1 was used in [10] and [11] in connection with the quantum cohomology ring of affine flag manifolds.

Here is an outline of the paper. We first prove the uniqueness part in the theorem, see Section 2. The main results are proved in Sections 3 and 4, where we consider the cases when Π is of one of the types F_4 or G_2 , respectively, and β is short. Finally, Appendix A deals with Theorem 1.1 in the case when β is the highest root, along the lines of Etingof's original proof from [3].

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2. UNIQUENESS OF $\Omega_1, \dots, \Omega_r$

We continue the notation of the introduction. Since β is a dominant root, in the expansion

$$(2.1) \quad \beta = m_1\alpha_1 + \dots + m_r\alpha_r$$

the coefficients m_i are all strictly positive integers. The following result is a direct consequence of [6, Lemma 3.6].

Proposition 2.1. ([6]) *If $\Gamma \in U(\mathfrak{b})_{\text{ev}}$ commutes with Ω then Γ is a multiple of $(X_0X_1^{m_1} \dots X_r^{m_r})^2$. In particular, if $\Gamma \neq 0$, then the degree of Γ is at least equal to $2 + 2 \sum_{i=1}^r m_i$.*

The table below contains the degrees of the fundamental W -invariant polynomials u_1, \dots, u_r and the sum of the coefficients m_1, \dots, m_r in equation (2.1). The information is extracted from [8, p. 477] and [6, p. 372].

Type	$\deg u_1, \dots, \deg u_r$	$\sum_{i=1}^r m_i$ (β long)	$\sum_{i=1}^r m_i$ (β short)
A_r ($r \geq 1$)	$2, 3, \dots, r+1$	r	r
B_r ($r \geq 2$)	$2, 4, 6, \dots, 2r$	$2r-1$	r
C_r ($r \geq 2$)	$2, 4, 6, \dots, 2r$	$2r-1$	$2r-2$
D_r ($r \geq 3$)	$2, 4, 6, \dots, 2r-2, r$	$2r-3$	$2r-3$
E_6	$2, 5, 6, 8, 9, 12$	11	11
E_7	$2, 6, 8, 10, 12, 14, 18$	17	17
E_8	$2, 8, 12, 14, 18, 20, 24, 30$	29	29
F_4	$2, 6, 8, 12$	11	8
G_2	$2, 6$	5	3

The table shows that $\deg u_1, \dots, \deg u_r$ are always at most equal to $2 \sum_{i=1}^r m_i$. Thus Proposition 2.1 immediately implies as follows.

Corollary 2.2. *For fixed $i \in \{1, \dots, r\}$, there exists at most one $\Omega_i \in U(\mathfrak{b})_{\text{ev}}$ such that $[\Omega_i, \Omega] = 0$, $\deg \Omega_i = \deg u_i$, and the symbol of Ω_i is equal to u_i .*

3. THE CASE WHEN Π IS OF TYPE F_4 AND β IS SHORT

We will prove Theorem 1.1 in the case mentioned in the title. The proof's main tool is the folding of the extended (untwisted) Dynkin diagram $E_7^{(1)}$ onto the extended twisted Dynkin diagram $E_6^{(2)}$. Figure 1 below is taken from [2, Section 3.3] and describes the folding map.

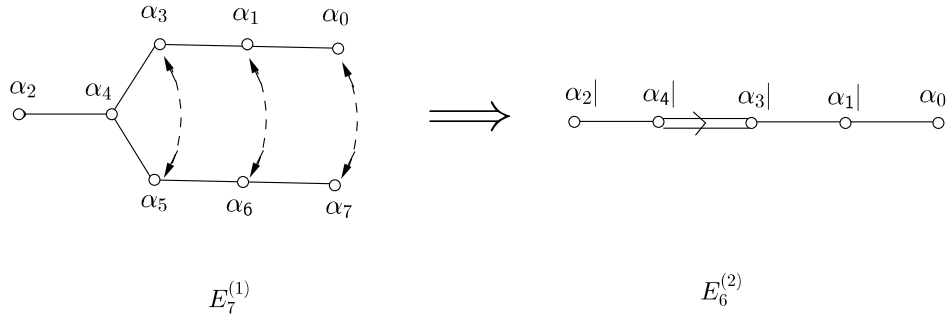


FIGURE 1. The folding $E_7^{(1)} \rightarrow E_6^{(2)}$.

More specifically, let $\Pi \subset \mathfrak{a}^*$ be the root system of type E_7 and pick simple roots $\{\alpha_1, \dots, \alpha_7\}$ fitting the diagram above. As usual, $\alpha_0 = -\theta$, where

$$\theta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$$

is the corresponding highest root. Consider the automorphism τ of \mathfrak{a}^* which is determined by

$$\tau(\alpha_1) = \alpha_6, \tau(\alpha_2) = \alpha_2, \tau(\alpha_3) = \alpha_5, \tau(\alpha_4) = \alpha_4, \tau(\alpha_5) = \alpha_3, \tau(\alpha_6) = \alpha_1, \tau(\alpha_7) = \alpha_0.$$

It follows that $\tau(\alpha_0) = \alpha_7$, hence τ is an automorphism of order two of \mathfrak{a}^* which permutes the elements of Π . Since \mathfrak{a} and \mathfrak{a}^* are canonically isomorphic, τ is also an automorphism of \mathfrak{a} . Let \mathfrak{a}' be the fixed point set of τ . Consider the restriction map $\mathfrak{a}^* \rightarrow (\mathfrak{a}')^*$ and denote the image of an arbitrary $\lambda \in \mathfrak{a}^*$ by $\lambda|$. The elements of Π are mapped to a root system of type F_4 , which we denote by $\Pi|$. A simple root system for $\Pi|$ is $\{\alpha_1|, \alpha_2|, \alpha_3|, \alpha_4|\}$. Moreover,

$$\theta| = 4\alpha_1| + 2\alpha_2| + 6\alpha_3| + 4\alpha_4| - \theta|$$

thus

$$\theta| = 2\alpha_1| + \alpha_2| + 3\alpha_3| + 2\alpha_4|,$$

which is the short dominant root of $\Pi|$ (see [6, p. 372]).

By the definition in Section 1, the “ $ax + b$ ” algebra associated to the extended Dynkin diagram $E_7^{(1)}$ is $\mathfrak{b} = \mathfrak{a} \oplus \text{Span}_{\mathbb{R}}\{X_0, \dots, X_7\}$, equipped with the Lie bracket which is identically zero on both \mathfrak{a} and $\text{Span}_{\mathbb{R}}\{X_0, \dots, X_7\}$ and such that

$$[H, X_i] = \alpha_i(H)X_i, \text{ for all } 0 \leq i \leq 7 \text{ and all } H \in \mathfrak{a}.$$

We now extend $\tau : \mathfrak{a} \rightarrow \mathfrak{a}$ to the linear automorphism $\tau : \mathfrak{b} \rightarrow \mathfrak{b}$ such that:

$$\begin{aligned} \tau(X_0) &= X_7, \quad \tau(X_1) = X_6, \quad \tau(X_2) = X_2, \quad \tau(X_3) = X_5, \\ \tau(X_4) &= X_4, \quad \tau(X_5) = X_3, \quad \tau(X_6) = X_1, \quad \tau(X_7) = X_0. \end{aligned}$$

Lemma 3.1. *The map $\tau : \mathfrak{b} \rightarrow \mathfrak{b}$ is a Lie algebra automorphism.*

Proof. One needs to check that $\tau([H, X_i]) = [\tau(H), \tau(X_i)]$, for all $H \in \mathfrak{a}$ and all $0 \leq i \leq 7$. We will confine ourselves to the case when $i = 0$. We have

$$\tau([H, X_0]) = \tau(\alpha_0(H)X_0) = \alpha_0(H)X_7,$$

whereas

$$[\tau(H), \tau(X_0)] = [\tau(H), X_7] = \alpha_7(\tau(H))X_7 = \alpha_0(H)X_7.$$

□

The eigenspace decomposition of τ is $\mathfrak{b} = \mathfrak{b}' \oplus \mathfrak{b}''$, where $\mathfrak{b}' := \{Z \in \mathfrak{b} \mid \tau(Z) = Z\}$ and $\mathfrak{b}'' = \{Z \in \mathfrak{b} \mid \tau(Z) = -Z\}$. Note that $X_0 + X_7, X_1 + X_6, X_3 + X_5, X_2, X_4 \in \mathfrak{b}'$ and for any $H' \in \mathfrak{a}'$ one has

$$\begin{aligned} [H', X_0 + X_7] &= \alpha_0(H')(X_0 + X_7) \\ [H', X_1 + X_6] &= \alpha_1(H')(X_1 + X_6) \\ [H', X_2] &= \alpha_2(H')X_2 \\ [H', X_3 + X_5] &= \alpha_3(H')(X_3 + X_5) \\ [H', X_4] &= \alpha_4(H')X_4. \end{aligned} \tag{3.1}$$

That is, $\mathfrak{b}' = \mathfrak{a}' \oplus \mathfrak{u}'$, where \mathfrak{u}' has a basis consisting of

$$X'_0 := \frac{1}{\sqrt{2}}(X_0 + X_7), X'_1 := \frac{1}{\sqrt{2}}(X_1 + X_6), X'_2 := X_2, X'_3 := \frac{1}{\sqrt{2}}(X_3 + X_5), X'_4 := X_4.$$

We have inserted the factors of $1/\sqrt{2}$ for later use, see equation (3.8) below. Equations (3.1) are telling us that \mathfrak{b}' is a Lie subalgebra of \mathfrak{b} isomorphic to the “ $ax + b$ ” algebra that corresponds to the Dynkin diagram $E_6^{(2)}$ (that is, F_4 extended with the short dominant root).

Denote by $\langle U(\mathfrak{b})\mathfrak{b}'' \rangle$ the linear span of $U(\mathfrak{b})\mathfrak{b}''$.

Lemma 3.2. *One has*

$$U(\mathfrak{b}) = U(\mathfrak{b}') \oplus \langle U(\mathfrak{b})\mathfrak{b}'' \rangle. \tag{3.2}$$

Proof. Pick bases Y_1, \dots, Y_9 of \mathfrak{b}' and Z_1, \dots, Z_6 of \mathfrak{b}'' . The corresponding PBW basis of $U(\mathfrak{b})$ consists of $Y^I Z^J$, where $I = (i_1, \dots, i_9)$ and $J = (j_1, \dots, j_6)$ are vectors with components in $\mathbb{Z}_{\geq 0}$. The PBW basis of $U(\mathfrak{b}')$ is $\{Y^I\}$, hence $U(\mathfrak{b}) = U(\mathfrak{b}') + \langle U(\mathfrak{b})\mathfrak{b}'' \rangle$. We now show that the latter sum is direct. To this end, note that if I and J are as before and $k \in \{1, \dots, 6\}$, then $Y^I Z^J Z_k$ is equal to $Y^I Z_1^{j_1} \dots Z_k^{j_k+1} \dots Z_6^{j_6}$ plus terms whose degree is strictly smaller than the degree of $Y^I Z^J Z_k$. Hence if $\Gamma \in \langle U(\mathfrak{b})\mathfrak{b}'' \rangle$ has degree m then in its PBW expansion the degree m monomials are of the form $Y^I Z^J$, where at least one of j_1, \dots, j_6 is non-zero; thus Γ cannot be in $U(\mathfrak{b}')$. \square

Denote by $\nu : U(\mathfrak{b}) \rightarrow U(\mathfrak{b}')$ the projection onto the first component of the decomposition (3.2). We will also need the eigenspace decomposition of $\tau|_{\mathfrak{a}}$, which is $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}''$; note that ν maps \mathfrak{a} onto \mathfrak{a}' as the orthogonal projection map.

Remark 3.3. The map $\nu : U(\mathfrak{b}) \rightarrow U(\mathfrak{b}')$ is not a homomorphism of (associative) algebras. More precisely, its restriction $\mathfrak{b} \rightarrow \mathfrak{b}'$ is not a homomorphism of Lie algebras. For $\nu(X_0) = \nu(X_7) = \frac{1}{2}(X_0 + X_7)$ and if $H \in \mathfrak{a}$ is such that $\alpha_0(H) = 0$ and $\alpha_7(H) = 1$ then

$$[H, X_0] = 0 \text{ and } [H, X_7] = X_7.$$

If ν were a homomorphism, then we would have simultaneously

$$[\nu(H), X_0 + X_7] = 0 \text{ and } [\nu(H), X_0 + X_7] = X_0 + X_7,$$

which is impossible.

Lemma 3.4. *The map ν transforms elements of $U(\mathfrak{b})_{\text{ev}}$ into elements of $U(\mathfrak{b}')_{\text{ev}}$.*

Proof. Start with the observation that an element of $U(\mathfrak{b})$ is in $U(\mathfrak{b})_{\text{ev}}$ if and only if it is fixed by each of the eight algebra automorphisms $\rho_0, \dots, \rho_7 : U(\mathfrak{b}) \rightarrow U(\mathfrak{b})$, where ρ_i is the identity map on \mathfrak{a} and

$$\rho_i(X_i) := -X_i, \quad \rho_i(X_j) := X_j, \text{ for } j \neq i.$$

Observe that $\rho_2 \circ \tau = \tau \circ \rho_2$, hence ρ_2 preserves the decomposition (3.2). The same holds for ρ_4 . Consider now $\rho_{07} := \rho_0 \circ \rho_7 = \rho_7 \circ \rho_0$, which is the identity on \mathfrak{a} and:

$$\rho_{07}(X_0) = -X_0, \quad \rho_{07}(X_7) = -X_7, \quad \rho_{07}(X_j) = X_j, \text{ for } j \neq 0, 7.$$

One has $\rho_{07} \circ \tau = \tau \circ \rho_{07}$, hence $\rho_{07}(U(\mathfrak{b}')) = U(\mathfrak{b}')$ and $\rho_{07}(\langle U(\mathfrak{b})\mathfrak{b}'' \rangle) = \langle U(\mathfrak{b})\mathfrak{b}'' \rangle$. In the same way one defines ρ_{16} and ρ_{35} and note that they both preserve the decomposition (3.2). The result stated in the lemma follows from the fact that if $\Gamma \in U(\mathfrak{b})_{\text{ev}}$ then $\rho_i(\Gamma) = \Gamma$, for all $0 \leq i \leq 7$, hence

$$\rho_2(\nu(\Gamma)) = \nu(\Gamma), \quad \rho_4(\nu(\Gamma)) = \nu(\Gamma), \quad \rho_{07}(\nu(\Gamma)) = \nu(\Gamma), \quad \rho_{16}(\nu(\Gamma)) = \nu(\Gamma), \text{ and } \rho_{35}(\nu(\Gamma)) = \nu(\Gamma).$$

This implies that $\nu(\Gamma) \in U(\mathfrak{b}')_{\text{ev}}$. \square

Recall now that the degrees of the fundamental homogeneous generators for $\text{Sym}(\mathfrak{a})^{W_{E_7}}$ and $\text{Sym}(\mathfrak{a}')^{W_{F_4}}$ are 2, 6, 8, 10, 12, 14, 18 and 2, 6, 8, 12 respectively, see the table in Section 2.

Here is the main result of this section:

Proposition 3.5. (a) *There exists a system u_1, \dots, u_7 of fundamental generators of $\text{Sym}(\mathfrak{a})^{W_{E_7}}$ of degrees 2, 6, 8, 10, 12, 14, 18 respectively, such that $\nu(u_1), \nu(u_2), \nu(u_3)$, and $\nu(u_5)$ are fundamental generators of $\text{Sym}(\mathfrak{a}')^{W_{F_4}}$.*

(b) *Let $\Omega \in U_2(\mathfrak{b})$ be the Laplacian and $\Omega_1, \dots, \Omega_7$ the elements of $U(\mathfrak{b})$ associated to u_1, \dots, u_7 by Proposition A.1. Then $\Omega' := \nu(\Omega)$ is a multiple of the Laplacian of \mathfrak{b}' . Furthermore, $\Omega'_1 := \nu(\Omega_1), \Omega'_2 := \nu(\Omega_2), \Omega'_3 := \nu(\Omega_3)$, and $\Omega'_5 := \nu(\Omega_5)$ have symbols equal to $\nu(u_1), \nu(u_2), \nu(u_3)$, and $\nu(u_5)$, respectively. They all lie in $U(\mathfrak{b}')_{\text{ev}}$ and commute with each other.*

In what follows we will prove Proposition 3.5. We start with a concrete presentation of the roots of type E_7 . Take $\mathfrak{a} = \mathbb{R}^7$; the positive roots are the following linear functions on \mathfrak{a} (as usual, x_i are the coordinate functions):

$$\begin{aligned} 2x_i, \quad 1 \leq i \leq 7; \quad & x_1 \pm x_2 \pm x_3 \pm x_4; \\ x_1 \pm x_2 \pm x_5 \pm x_6; \quad & x_1 \pm x_3 \pm x_5 \pm x_7; \\ x_1 \pm x_4 \pm x_6 \pm x_7; \quad & x_2 \pm x_3 \pm x_6 \pm x_7; \\ x_2 \pm x_4 \pm x_5 \pm x_7; \quad & x_3 \pm x_4 \pm x_5 \pm x_6. \end{aligned}$$

A simple root system that fits the Dynkin diagram in Figure 1 is:

$$\begin{aligned} \alpha_1 &= x_2 - x_3 + x_6 - x_7, \quad \alpha_2 = x_1 - x_2 - x_3 - x_4, \quad \alpha_3 = x_3 - x_4 + x_5 - x_6, \\ \alpha_4 &= 2x_4, \quad \alpha_5 = x_3 - x_4 - x_5 + x_6, \quad \alpha_6 = x_2 - x_3 - x_6 + x_7, \quad \alpha_7 = -x_1 - x_2 - x_5 - x_6. \end{aligned}$$

The resulting highest root is $\theta = x_1 + x_2 - x_5 - x_6$. The involution τ is the linear transformation

$$e_i \mapsto e_i, \quad 1 \leq i \leq 4; \quad e_j \mapsto -e_j, \quad 5 \leq j \leq 7,$$

where e_1, \dots, e_7 is the canonical basis of \mathbb{R}^7 ; thus $\mathfrak{a}' = \mathbb{R}^4$. The restricted root system is of type F_4 and admits the following simple roots:

$$|\alpha_2| = x_1 - x_2 - x_3 - x_4, \quad |\alpha_4| = 2x_4, \quad |\alpha_3| = x_3 - x_4, \quad |\alpha_1| = x_2 - x_3.$$

According to [12, p. 1096], the following is a system of fundamental W_{E_7} -invariant elements of $\text{Sym}(\mathfrak{a})$:

$$\begin{aligned} (3.3) \quad v_k &:= \sum_{\pm} (e_1 \pm e_2 \pm e_7)^k + \sum_{\pm} (e_1 \pm e_3 \pm e_6)^k + \sum_{\pm} (e_1 \pm e_4 \pm e_5)^k + \sum_{\pm} (e_2 \pm e_3 \pm e_5)^k \\ &+ \sum_{\pm} (e_2 \pm e_4 \pm e_6)^k + \sum_{\pm} (e_3 \pm e_4 \pm e_7)^k + \sum_{\pm} (e_5 \pm e_6 \pm e_7)^k, \end{aligned}$$

where $k = 2, 6, 8, 10, 12, 14, 18$ (here Σ_{\pm} denotes the sum over all four possible choices of the two signs). For $k = 2, 6, 8, 12$, the projection $\nu : \text{Sym}(\mathfrak{a}) \rightarrow \text{Sym}(\mathfrak{a}')$ maps the expressions above to

$$(3.4) \quad \nu(v_k) = 2 \sum_{1 \leq i < j \leq 4} (e_i - e_j)^k + (e_i + e_j)^k, \quad k = 2, 6, 8, 12,$$

which are a system of fundamental W_{F_4} -invariant elements of $\text{Sym}(\mathfrak{a}')$, see [12, Section 2.4]. In this way we have proved point (a) of Proposition 3.5.

We now prove point (b). One has

$$\Omega = u_1 + X_0^2 + \cdots + X_7^2, \quad \text{where } u_1 = v_2.$$

Observe that

$$(3.5) \quad X_0^2 + X_7^2 = \frac{1}{2}(X_0 + X_7)^2 + \frac{1}{2}(X_0 - X_7)^2 = (X'_0)^2 + \frac{1}{2}(X_0 - X_7)^2$$

$$(3.6) \quad X_1^2 + X_6^2 = \frac{1}{2}(X_1 + X_6)^2 + \frac{1}{2}(X_1 - X_6)^2 = (X'_1)^2 + \frac{1}{2}(X_1 - X_6)^2$$

$$(3.7) \quad X_3^2 + X_5^2 = \frac{1}{2}(X_3 + X_5)^2 + \frac{1}{2}(X_3 - X_5)^2 = (X'_3)^2 + \frac{1}{2}(X_3 - X_5)^2.$$

Since $X_0 - X_7, X_1 - X_6$, and $X_3 - X_5$ are in \mathfrak{b}'' , we deduce that

$$(3.8) \quad \nu(\Omega_1) = \nu(u_1) + (X'_0)^2 + (X'_1)^2 + (X'_2)^2 + (X'_3)^2 + (X'_4)^2.$$

By equation (1.1), this is a scalar multiple of the Laplacian of \mathfrak{b}' (the reason is that $\nu(u_1)$ is a scalar multiple of $\sum_{i,j=1}^4 \langle \alpha_i |, \alpha_j | \rangle H'_i H'_j$, where $H'_1, \dots, H'_4 \in \mathfrak{a}'$ with $\alpha_i(H'_j) = \delta_{ij}$, $1 \leq i, j \leq 4$; one may need to rescale the metric on \mathfrak{u}' such that $\{X'_0, X'_1, X'_2, X'_3, X'_4\}$ is an orthonormal basis).

Let $\tau : U(\mathfrak{b}) \rightarrow U(\mathfrak{b})$ be the algebra automorphism induced by Lemma 3.1.

Lemma 3.6. (a) If $\Gamma \in U(\mathfrak{b}')$ then $\tau(\Gamma) = \Gamma$.

(b) If $\Gamma \in U(\mathfrak{b})$ has the property that $\tau(\Gamma) = -\Gamma$, then $\Gamma \in \langle U(\mathfrak{b})\mathfrak{b}'' \rangle$.

Proof. Only (b) needs to be justified. By Lemma 3.2, one can write $\Gamma = \Gamma' + \Gamma''$, where $\Gamma' \in U(\mathfrak{b}')$ and $\Gamma'' \in \langle U(\mathfrak{b})\mathfrak{b}'' \rangle$. This implies that $\tau(\Gamma) = \Gamma' + \tau(\Gamma'')$. Since $\tau(\Gamma'') \in \langle U(\mathfrak{b})\mathfrak{b}'' \rangle$, one has $\Gamma' = 0$. \square

It now comes the last step in the proof of Proposition 3.5.

Lemma 3.7. (a) For any $i \in \{1, 2, 3, 5\}$, the symbol of $\nu(\Omega_i)$ in $U(\mathfrak{b}')$ is $\nu(u_i)$.

(b) For any two $i, j \in \{1, 2, 3, 5\}$ one has $[\nu(\Omega_i), \nu(\Omega_j)] = 0$.

Proof. (a) Let $\mu' : U(\mathfrak{b}') \rightarrow U(\mathfrak{a}')$ be the symbol map. It is sufficient to show that the following diagram is commutative.

$$\begin{array}{ccc} U(\mathfrak{b}) & \xrightarrow{\mu} & U(\mathfrak{a}) \\ \downarrow \nu & & \downarrow \nu \\ U(\mathfrak{b}') & \xrightarrow{\mu'} & U(\mathfrak{a}') \end{array}$$

Take $\Gamma \in U(\mathfrak{b})$ and decompose it as $\Gamma = \Gamma' + \sum_{k=1}^p a_k B_k$ where $\Gamma' \in U(\mathfrak{b}')$, $a_k \in U(\mathfrak{b})$, and $B_k \in \mathfrak{b}''$. Both $\nu : \mathfrak{a} \rightarrow \mathfrak{a}'$ and $\mu : \mathfrak{b} \rightarrow \mathfrak{a}$ are Lie algebra homomorphisms, hence

$\nu \circ \mu(\sum_{k=1}^p a_k B_k) = \sum_{k=1}^p \nu(\mu(a_k))\nu(\mu(B_k)) = 0$, since $\nu(\mu(B_k)) = 0$ (recall that $\mathfrak{b}'' = \mathfrak{a}'' \oplus \text{Span}\{X_0 - X_7, X_1 - X_6, X_3 - X_5\}$). It remains to show that $\nu \circ \mu(\Gamma') = \mu'(\Gamma')$. But this follows readily from the fact that $\nu \circ \mu|_{\mathfrak{b}'} : \mathfrak{b}' \rightarrow \mathfrak{a}'$ is equal to μ' , the map being a Lie algebra homomorphism.

(b) Decompose

$$\begin{aligned}\Omega_i &= \nu(\Omega_i) + \Omega_i'' \\ \Omega_j &= \nu(\Omega_j) + \Omega_j'',\end{aligned}$$

where $\Omega_i'', \Omega_j'' \in \langle U(\mathfrak{b})\mathfrak{b}'' \rangle$. From $\Omega_i\Omega_j = \Omega_j\Omega_i$ one obtains:

$$\begin{aligned}(3.9) \quad & \nu(\Omega_i)\nu(\Omega_j) + \nu(\Omega_i)\Omega_j'' + \Omega_i''\nu(\Omega_j) + \Omega_i''\Omega_j'' \\ &= \nu(\Omega_j)\nu(\Omega_i) + \nu(\Omega_j)\Omega_i'' + \Omega_j''\nu(\Omega_i) + \Omega_j''\Omega_i''.\end{aligned}$$

We claim that except $\nu(\Omega_i)\nu(\Omega_j)$ and $\nu(\Omega_j)\nu(\Omega_i)$, all terms in both sides of the equation above are in $\langle U(\mathfrak{b})\mathfrak{b}'' \rangle$. The claim is clearly true for $\nu(\Omega_i)\Omega_j'', \nu(\Omega_j)\Omega_i'', \Omega_i''\Omega_j''$, and $\Omega_j''\Omega_i''$. To show that $\Omega_j''\nu(\Omega_i)$ is in $\langle U(\mathfrak{b})\mathfrak{b}'' \rangle$, we note that $\Omega_j'' = \sum_{k=1}^p a_k B_k$ for some $a_k \in U(\mathfrak{b})$ and $B_k \in \mathfrak{b}''$. Hence $\Omega_j''\nu(\Omega_i) = \sum_{k=1}^p a_k B_k \nu(\Omega_i)$. But by Lemma 3.6 (a), for any $1 \leq k \leq p$ we have $\tau(B_k \nu(\Omega_i)) = \tau(B_k)\tau(\nu(\Omega_i)) = -B_k \nu(\Omega_i)$; hence, by Lemma 3.6 (b), the product $B_k \nu(\Omega_i)$ is in $\langle U(\mathfrak{b})\mathfrak{b}'' \rangle$. By multiplying each such product from the left by a_k and adding up all these expressions ($1 \leq k \leq p$), the result will be in $\langle U(\mathfrak{b})\mathfrak{b}'' \rangle$. \square

As a consequence of Proposition 3.5, we have:

Corollary 3.8. *The conclusion of Theorem 1.1 is true in the case when Π is of type F_4 and β is the short dominant root.*

Proof. Proposition 3.5 is telling us that the conclusion of Theorem 1.1 is true for a special choice of the fundamental W_{F_4} -invariant homogeneous polynomials, namely $\nu(u_1), \nu(u_2), \nu(u_3)$, and $\nu(u_5)$. Let now $u'_1, u'_2, u'_3, u'_4 \in S(\mathfrak{a}')^{W_{F_4}}$ be an arbitrary system of such polynomials. For each $j = 1, 2, 3, 4$, one can write u'_j as $f_j(\nu(u_1), \nu(u_2), \nu(u_3), \nu(u_5))$, where f_j is a polynomial in four variables. Then $\Omega'_j := f_j(\nu(\Omega_1), \nu(\Omega_2), \nu(\Omega_3), \nu(\Omega_5))$, where $1 \leq j \leq 4$, satisfy the conditions in Proposition 3.5 relative to u'_1, u'_2, u'_3, u'_4 . \square

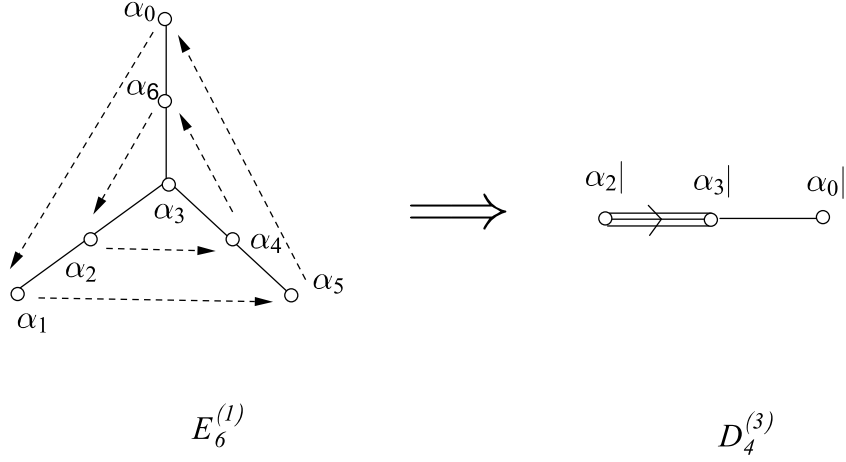
4. THE CASE WHEN Π IS OF TYPE G_2 AND β IS SHORT

In the case mentioned in the title, we will use the same approach as in the previous section, this time for the folding of the extended Dynkin diagram $E_6^{(1)}$ onto the extended twisted Dynkin diagram $D_4^{(3)}$. This is described in Figure 2 below, see [2, Section 3.4]. Here $\{\alpha_1, \dots, \alpha_6\} \subset \mathfrak{a}^*$ are simple roots of the root system Π of type E_6 , and $\alpha_0 = -\theta$ is the negative of the highest root. One has

$$\theta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6.$$

Let τ be the automorphism of \mathfrak{a}^* determined by

$$\tau(\alpha_1) = \alpha_5, \tau(\alpha_2) = \alpha_4, \tau(\alpha_3) = \alpha_3, \tau(\alpha_4) = \alpha_6, \tau(\alpha_5) = \alpha_0, \tau(\alpha_6) = \alpha_2.$$

FIGURE 2. The folding $E_6^{(1)} \rightarrow D_4^{(3)}$.

One deduces that $\tau(\alpha_0) = \alpha_1$, hence τ is a diagram automorphism of order three, that is, $\tau^3 = \text{id}$. The canonical isomorphism $\mathfrak{a}^* \simeq \mathfrak{a}$ induces an automorphism τ of \mathfrak{a} . Let $\mathfrak{a}' \subset \mathfrak{a}$ be the fixed point set of τ . For $\lambda \in \mathfrak{a}^*$, we denote by $\lambda|$ the restriction to \mathfrak{a}' . Then $\Pi| := \{\alpha| ; \alpha \in \Pi\}$ is a root system of type G_2 . It admits $\{\alpha_2|, \alpha_3|\}$ as simple roots. One has $\theta| = 2\alpha_2| + \alpha_3|$, which is the short dominant root of $\Pi|$ (see [6, p. 372]).

The “ $ax + b$ ” algebra of the extended Dynkin diagram $E_6^{(1)}$ is $\mathfrak{b} = \mathfrak{a} \oplus \text{Span}_{\mathbb{R}}\{X_0, \dots, X_6\}$ where

$$[H, X_i] = \alpha_i(H)X_i, \text{ for all } 0 \leq i \leq 6 \text{ and all } H \in \mathfrak{a}.$$

Let $\tau : \mathfrak{b} \rightarrow \mathfrak{b}$ be the linear automorphism which extends the previous τ such that

$$\begin{aligned} \tau(X_0) &= X_1, \tau(X_1) = X_5, \tau(X_2) = X_4, \tau(X_3) = X_3, \\ \tau(X_4) &= X_6, \tau(X_5) = X_0, \tau(X_6) = X_2. \end{aligned}$$

One shows that τ is a Lie algebra automorphism, by proceeding in the same way as in Lemma 3.1.

Let $\mathfrak{b}' \subset \mathfrak{b}$ be the fixed point set of τ . Observe that $X_0 + X_1 + X_5, X_2 + X_4 + X_6$, and X_3 are in \mathfrak{b}' and that for any $H' \in \mathfrak{a}'$, one has

$$\begin{aligned} [H', X_0 + X_1 + X_5] &= \alpha_0(H')(X_0 + X_1 + X_5), \\ [H', X_2 + X_4 + X_6] &= \alpha_2(H')(X_2 + X_4 + X_6), \\ [H', X_3] &= \alpha_3(H')X_3. \end{aligned} \tag{4.1}$$

That is, $\mathfrak{b}' = \mathfrak{a}' \oplus \mathfrak{u}'$, where \mathfrak{u}' has a basis consisting of

$$X'_0 := \frac{1}{\sqrt{3}}(X_0 + X_1 + X_5), X'_2 := \frac{1}{\sqrt{3}}(X_2 + X_4 + X_6), X'_3 := X_3.$$

By (4.1) \mathfrak{b}' is isomorphic to the “ $ax + b$ ” algebra that corresponds to the Dynkin diagram $D_4^{(3)}$.

Let $\mathfrak{b}_{\mathbb{C}}$ and $\mathfrak{b}'_{\mathbb{C}}$ be the complexifications of \mathfrak{b} and \mathfrak{b}' . Extend τ by \mathbb{C} -linearity to an automorphism of $\mathfrak{b}_{\mathbb{C}}$. Set $\varepsilon := \cos(\pi/3) + i \sin(\pi/3)$. The eigenspace decomposition of τ is $\mathfrak{b}_{\mathbb{C}} = \mathfrak{b}'_{\mathbb{C}} \oplus \mathfrak{b}''$, where

$$\mathfrak{b}'' = \{Z \in \mathfrak{b}_{\mathbb{C}} \mid \tau(Z) = \varepsilon Z\} \oplus \{Z \in \mathfrak{b}_{\mathbb{C}} \mid \tau(Z) = \varepsilon^2 Z\}.$$

The following result can be proved with the same method as Lemma 3.2.

Lemma 4.1. *One has*

$$(4.2) \quad U(\mathfrak{b}_{\mathbb{C}}) = U(\mathfrak{b}'_{\mathbb{C}}) \oplus \langle U(\mathfrak{b}_{\mathbb{C}})\mathfrak{b}'' \rangle,$$

where $\langle U(\mathfrak{b}_{\mathbb{C}})\mathfrak{b}'' \rangle$ is the \mathbb{C} -linear span of $U(\mathfrak{b}_{\mathbb{C}})\mathfrak{b}''$.

We denote by $\nu : U(\mathfrak{b}_{\mathbb{C}}) \rightarrow U(\mathfrak{b}'_{\mathbb{C}})$ the projection onto the first component of the decomposition (4.2). Observe that ν maps \mathfrak{a} to \mathfrak{a}' as the orthogonal projection map.

In the same way as Lemma 3.4, we have:

Lemma 4.2. *The map ν transforms elements of $U(\mathfrak{b}_{\mathbb{C}})_{\text{ev}}$ into elements of $U(\mathfrak{b}'_{\mathbb{C}})_{\text{ev}}$.*

From the table in Section 2, the degrees of the fundamental homogeneous generators for $\text{Sym}(\mathfrak{a})^{W_{E_6}}$ are 2, 5, 6, 8, 9, and 12, whereas for $\text{Sym}(\mathfrak{a}')^{W_{G_2}}$ they are 2 and 6. We will prove as follows:

Proposition 4.3. (a) *There exists a system u_1, \dots, u_6 of fundamental generators of $\text{Sym}(\mathfrak{a})^{W_{E_6}}$ of degrees 2, 5, 6, 8, 9, and 12 respectively, such that $\nu(u_1)$ and $\nu(u_3)$ are fundamental generators of $\text{Sym}(\mathfrak{a}')^{W_{G_2}}$.*

(b) *Let $\Omega \in U_2(\mathfrak{b})$ be the Laplacian and $\Omega_1, \dots, \Omega_6$ the elements of $U(\mathfrak{b})$ associated to u_1, \dots, u_6 by Proposition A.1. Then $\Omega' := \nu(\Omega)$ is a multiple of the Laplacian of \mathfrak{b}' . Furthermore, $\Omega'_3 := \nu(\Omega_3)$ belongs to $U(\mathfrak{b}'_{\mathbb{C}})_{\text{ev}}$, has its symbol equal to $\nu(u_3)$, and commutes with Ω' .*

To prove the proposition, we need the following presentation of the root system of type E_6 , see [12, p. 1095]. One takes $\mathfrak{a} = \mathbb{R}^6$. The positive roots are:

$$\begin{aligned} & 2x_i, \quad 1 \leq i \leq 4; \quad x_1 \pm x_2 \pm x_3 \pm x_4; \\ & x_1 \pm x_2 \pm \sqrt{2}x_5; \quad x_3 \pm x_4 \pm \sqrt{2}x_5; \\ & x_1 \pm x_3 \pm \frac{1}{\sqrt{2}}(x_5 - \sqrt{3}x_6); \quad x_2 \pm x_4 \pm \frac{1}{\sqrt{2}}(x_5 - \sqrt{3}x_6); \\ & x_1 \pm x_4 \pm \frac{1}{\sqrt{2}}(x_5 + \sqrt{3}x_6); \quad x_2 \pm x_3 \pm \frac{1}{\sqrt{2}}(x_5 + \sqrt{3}x_6). \end{aligned}$$

A simple root system is:

$$\begin{aligned}\alpha_1 &= x_2 - x_3 - \frac{1}{\sqrt{2}}(x_5 + \sqrt{3}x_6), & \alpha_2 &= x_3 - x_4 + \sqrt{2}x_5, & \alpha_3 &= 2x_4, \\ \alpha_4 &= x_3 - x_4 - \sqrt{2}x_5, & \alpha_5 &= x_2 - x_3 + \frac{1}{\sqrt{2}}(x_5 + \sqrt{3}x_6), & \alpha_6 &= x_1 - x_2 - x_3 - x_4.\end{aligned}$$

The highest root is

$$\theta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 = 2x_1.$$

Thus $\alpha_0 = -2x_1$.

The subspace \mathfrak{a}' of \mathfrak{a} is described by the equations $\alpha_2 = \alpha_4 = \alpha_6$ and $\alpha_0 = \alpha_1 = \alpha_5$; they are equivalent to:

$$x_1 = -x_2 = x_3, \quad x_5 = x_6 = 0.$$

A simple root system for $\Pi|$ consists of the following two linear functions on \mathfrak{a}' :

$$\alpha_2| = x_1 - x_4 \text{ and } \alpha_3| = 2x_4.$$

The short positive roots are $x_1 - x_4$, $-2x_1$, and $-x_1 - x_4$.

The following is a system of fundamental W_{E_6} -invariant elements of $\text{Sym}(\mathfrak{a}^*)$ (see [12, p. 1096]):

$$\begin{aligned}v_k &:= \left(2\sqrt{\frac{2}{3}}x_6\right)^k + \left(\sqrt{\frac{2}{3}}(\sqrt{3}x_5 - x_6)\right)^k + \left(\sqrt{\frac{2}{3}}(-\sqrt{3}x_5 - x_6)\right)^k \\ &+ \sum_{\pm} \left(\pm x_3 \pm x_4 - \sqrt{\frac{2}{3}}x_6\right)^k + \sum_{\pm} \left(\pm x_1 \pm x_2 - \sqrt{\frac{2}{3}}x_6\right)^k \\ &+ \sum_{\pm} \left(\pm x_2 \pm x_4 + \frac{1}{\sqrt{6}}(\sqrt{3}x_5 + x_6)\right)^k + \sum_{\pm} \left(\pm x_1 \pm x_3 + \frac{1}{\sqrt{6}}(\sqrt{3}x_5 + x_6)\right)^k \\ &+ \sum_{\pm} \left(\pm x_2 \pm x_3 - \frac{1}{\sqrt{6}}(\sqrt{3}x_5 - x_6)\right)^k + \sum_{\pm} \left(\pm x_1 \pm x_4 - \frac{1}{\sqrt{6}}(\sqrt{3}x_5 + x_6)\right)^k,\end{aligned}$$

where $k = 2, 5, 6, 8, 9, 12$ (here Σ_{\pm} denotes the sum over all four possible choices of the two signs). We are particularly interested in the cases $k = 2$ and $k = 6$. The images of v_2 and v_6 by the restriction map $\nu : \text{Sym}(\mathfrak{a}^*) \rightarrow \text{Sym}((\mathfrak{a}')^*)$ are

$$\begin{aligned}\nu(v_2) &= 6 \left[(x_1 - x_4)^2 + (2x_1)^2 + (x_1 + x_4)^2 \right], \\ \nu(v_6) &= 6 \left[(x_1 - x_4)^6 + (2x_1)^6 + (x_1 + x_4)^6 \right].\end{aligned}$$

Since W_{G_2} permutes the short roots, the two polynomials above are W_{G_2} -invariant. They are a system of fundamental invariant polynomials: the reason is that the degrees of any two such polynomials are 2 and 6 and $\nu(v_6)$ is not a scalar multiple of $(\nu(v_2))^3$. Since otherwise we would have

$$(3x_1^2 + x_4^2)^3 = r \left[(x_1 - x_4)^6 + (2x_1)^6 + (x_1 + x_4)^6 \right]$$

for some $r \in \mathbb{R}$. By making $x_1 = 0$ one obtains $1 = r \cdot 2$ and by making $x_4 = 0$ one obtains $27 = r \cdot 66$, which is a contradiction. One now identifies $\mathfrak{a} = \mathfrak{a}^*$ and $\mathfrak{a}' = (\mathfrak{a}')^*$ via the canonical inner product on $\mathfrak{a} = \mathbb{R}^6$ and concludes that point (a) of Proposition 4.3 holds true.

From now on we regard again v_2 and v_6 as elements of $\text{Sym}(\mathfrak{a})$ and $\text{Sym}(\mathfrak{a}')$. That is,

$$(4.3) \quad \begin{aligned} v_k := & \left(2\sqrt{\frac{2}{3}}e_6\right)^k + \left(\sqrt{\frac{2}{3}}(\sqrt{3}e_5 - e_6)\right)^k + \left(\sqrt{\frac{2}{3}}(-\sqrt{3}e_5 - e_6)\right)^k \\ & + \sum_{\pm} \left(\pm e_3 \pm e_4 - \sqrt{\frac{2}{3}}e_6\right)^k + \sum_{\pm} \left(\pm e_1 \pm e_2 - \sqrt{\frac{2}{3}}e_6\right)^k \\ & + \sum_{\pm} \left(\pm e_2 \pm e_4 + \frac{1}{\sqrt{6}}(\sqrt{3}e_5 + e_6)\right)^k + \sum_{\pm} \left(\pm e_1 \pm e_3 + \frac{1}{\sqrt{6}}(\sqrt{3}e_5 + e_6)\right)^k \\ & + \sum_{\pm} \left(\pm e_2 \pm e_3 - \frac{1}{\sqrt{6}}(\sqrt{3}e_5 - e_6)\right)^k + \sum_{\pm} \left(\pm e_1 \pm e_4 - \frac{1}{\sqrt{6}}(\sqrt{3}e_5 + e_6)\right)^k, \end{aligned}$$

for $k = 2, 6$.

Let $\Omega_1, \dots, \Omega_6$ be the elements of $U(\mathfrak{b})$ described in Theorem 1.1 for Π of type E_6 (see Appendix A). One has

$$\Omega_1 = u_1 + X_0^2 + \dots + X_6^2, \quad \text{where } u_1 = v_2.$$

Observe that

$$\begin{aligned} X_0^2 + X_1^2 + X_5^2 &= \frac{1}{3}(X_0 + X_1 + X_5)^2 + \frac{1}{3}(X_0 - X_1)^2 + \frac{1}{3}(X_1 - X_5)^2 + \frac{1}{3}(X_0 - X_5)^2, \\ X_2^2 + X_4^2 + X_6^2 &= \frac{1}{3}(X_2 + X_4 + X_6)^2 + \frac{1}{3}(X_2 - X_4)^2 + \frac{1}{3}(X_4 - X_6)^2 + \frac{1}{3}(X_6 - X_2)^2. \end{aligned}$$

Lemma 4.4. *The differences $X_0 - X_1, X_1 - X_5, X_0 - X_5, X_2 - X_4, X_4 - X_6$, and $X_6 - X_2$ are in \mathfrak{b}'' .*

Proof. Both $X_0 + \varepsilon X_1 + \varepsilon^2 X_5$ and $X_1 + \varepsilon X_0 + \varepsilon^2 X_5$ are in \mathfrak{b}'' and their difference is $(1 - \varepsilon)(X_0 - X_1)$. \square

This implies that

$$(4.4) \quad \nu(\Omega_1) = \nu(u_1) + (X'_0)^2 + (X'_2)^2 + (X'_3)^2,$$

which is a multiple of the Laplacian of \mathfrak{b}' .

Since $\tau : \mathfrak{b} \rightarrow \mathfrak{b}$ is a Lie algebra homomorphism, it induces an algebra homomorphism $\tau : U(\mathfrak{b}_{\mathbb{C}}) \rightarrow U(\mathfrak{b}_{\mathbb{C}})$.

Lemma 4.5. (a) *If $a \in U(\mathfrak{b}'_{\mathbb{C}})$ then $\tau(a) = a$.*

(b) *If $a \in U(\mathfrak{b}_{\mathbb{C}})$ has the property that $\tau(a) = \varepsilon a$ or $\tau(a) = \varepsilon^2 a$ then $a \in \langle U(\mathfrak{b}_{\mathbb{C}})\mathfrak{b}'' \rangle$.*

Proposition 4.3 can now be proved by the same argument as the one used at the end of Section 3 when proving Proposition 3.5.

From Proposition 4.3 we now deduce:

Corollary 4.6. *The conclusion of Theorem 1.1 is true in the case when Π is of type G_2 and β is the short dominant root.*

This can be proved in the same way as Corollary 3.8. The only essential difference is that Ω'_3 belongs to $U(\mathfrak{b}'_{\mathbb{C}})_{\text{ev}}$ and not necessarily to $U(\mathfrak{b}')_{\text{ev}}$. However, one can construct out of it an element of $U(\mathfrak{b}')_{\text{ev}}$ whose symbol is $\nu(u_3)$, has degree equal to 6, and which commutes with Ω' . Concretely, consider the basis of \mathfrak{b}' which consists of X'_0, X'_2, X'_3 along with $H'_2, H'_3 \in \mathfrak{a}'$ such that $\alpha_i(H'_j) = \delta_{ij}$, $i, j = 2$ or 3 . Relative to the induced PBW basis, $U(\mathfrak{b}'_{\mathbb{C}})_{\text{ev}} = U(\mathfrak{b}')_{\text{ev}} \oplus \sqrt{-1}U(\mathfrak{b}')_{\text{ev}}$. The aforementioned element of $U(\mathfrak{b}')_{\text{ev}}$ is just the first term in the decomposition of Ω'_3 relative to this splitting.

APPENDIX A. THE CASE WHEN β IS THE HIGHEST ROOT: THE THEOREM OF ETINGOF

The complete integrability of the periodic quantum Toda lattice corresponding to a Dynkin diagram extended by the highest root was proved by Etingof in [3]. We considered necessary to include the details of his proof for reasons of completeness and clarity. The approach below is slightly different from the original one, in that we preferred to use complex Lie groups and loop spaces rather than formal groups. As background references we indicate [9], [14], and [13]. Let G be a simple, simply connected, complex Lie group of Lie algebra \mathfrak{g} and $T \subset G$ a maximal torus, whose Lie algebra we denote by \mathfrak{h} . Let also $\langle \cdot, \cdot \rangle$ be the Killing form on \mathfrak{g} normalized such that $\langle \alpha, \alpha \rangle = 2$ for any long root α . Pick a simple root system $\alpha_1, \dots, \alpha_r \in \mathfrak{h}^*$ and let θ be the corresponding highest root. Consider the differential operator on \mathfrak{h} given by

$$M = \frac{1}{2}\Delta - Ke^{-\theta} - \sum_{i=1}^r e^{\alpha_i}$$

where Δ is the Laplacian relative to $\langle \cdot, \cdot \rangle$ and K is a parameter. The goal is to construct differential operators on \mathfrak{h} which commute with M . Recall [9] that the corresponding (non-twisted) affine Kac-Moody Lie algebra is $\hat{\mathfrak{g}} = \mathcal{L}_{\text{pol}}(\mathfrak{g}) \oplus \mathbb{C}c$, where $\mathcal{L}_{\text{pol}}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ is the space of Laurent polynomials with coefficients in \mathfrak{g} , z being on the unit circle S^1 . The Lie bracket on $\hat{\mathfrak{g}}$ is defined by

$$[u, v](z) = [u(z), v(z)] + (\text{Res}\langle u'(z), v(z) \rangle) c$$

for any $u, v \in \mathcal{L}_{\text{pol}}(\mathfrak{g})$, where Res stands for residue (the coefficient of z^{-1}) and c is a central element.

The Chevalley generators of $\hat{\mathfrak{g}}$ are e_i, h_i, f_i , where $0 \leq i \leq r$. Here $\{h_i\}$ is the basis of $\hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}c$ consisting of the simple affine coroots, $\{e_i\}$ are root vectors for simple affine

roots and $\{f_i\}$ root vectors for the negatives of those roots. Denote by \mathfrak{n}^+ and \mathfrak{n}^- the Lie subalgebras of $\hat{\mathfrak{g}}$ generated by $\{e_i\}$ and $\{f_i\}$ respectively. They can be described as:

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \hat{R}^+} \hat{\mathfrak{g}}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \hat{R}^-} \hat{\mathfrak{g}}_\alpha,$$

where the first sum runs over all positive affine roots and the second sum over all negative affine roots, $\hat{\mathfrak{g}}_\alpha$ being the corresponding root space. We thus have the triangular decomposition $\hat{\mathfrak{g}} = \mathfrak{n}^+ \oplus \hat{\mathfrak{h}} \oplus \mathfrak{n}^-$.

Let us now consider the group $\mathcal{L}(G)$ of all smooth maps from the circle S^1 to G along with its universal central extension $\tilde{\mathcal{L}}(G)$, see [14, Section 4.4]. Both $\mathcal{L}(G)$ and $\tilde{\mathcal{L}}(G)$ are Fréchet-Lie groups, of Lie algebras $\mathcal{L}(\mathfrak{g})$ (space of all smooth maps $S^1 \rightarrow \mathfrak{g}$) and $\mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c$, respectively. There is a well-defined exponential map $\exp : \tilde{\mathcal{L}}(\mathfrak{g}) \rightarrow \tilde{\mathcal{L}}(G)$, which is smooth.

Let $U^+, U^- \subset G$ be the unipotent radicals of the two standard “opposite” Borel subgroups that contain T . Denote by \mathcal{U}^+ the subgroup of $\mathcal{L}(G)$ whose elements are smooth boundary values of holomorphic maps $\gamma : \{z \in \mathbb{C} \mid |z| < 1\} \rightarrow G$ with $\gamma(0) \in U^+$. Similarly, \mathcal{U}^- is the subgroup of $\mathcal{L}(G)$ consisting of smooth boundary values of holomorphic maps $\gamma : \{z \in \mathbb{C} \mid |z| > 1\} \cup \{\infty\} \rightarrow G$ such that $\gamma(\infty) \in U^-$. There exist canonical embeddings $\mathcal{U}^\pm \subset \tilde{\mathcal{L}}(G)$. Moreover, if $\mathcal{T} := \exp(\hat{\mathfrak{h}}) = T \times \mathbb{C}^*$, then the map $\mathcal{U}^+ \times \mathcal{T} \times \mathcal{U}^- \rightarrow \tilde{\mathcal{L}}(G)$, $(u^+, g, u^-) \mapsto u^+ g u^-$ is injective onto $\mathcal{U}^+ \mathcal{T} \mathcal{U}^-$, which is an open subspace of $\tilde{\mathcal{L}}(G)$, see [14, Theorem 8.7.2]. Any element of $\hat{\mathfrak{g}}$ induces a canonical left-invariant vector field on $\tilde{\mathcal{L}}(G)$, hence also on its open subspace $\mathcal{U}^+ \mathcal{T} \mathcal{U}^-$. One obtains in this way a representation of the universal enveloping algebra $U(\hat{\mathfrak{g}})$ on $C^\infty(\mathcal{U}^+ \mathcal{T} \mathcal{U}^-, \mathbb{C})$. Concretely, for $\xi \in \hat{\mathfrak{g}}$, $u^\pm \in \mathcal{U}^\pm$, $g \in \mathcal{T}$, and $f : \mathcal{U}^+ \mathcal{T} \mathcal{U}^- \rightarrow \mathbb{C}$ smooth,

$$(A.1) \quad (\xi.f)(u^+ g u^-) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(t\xi) u^+ g u^-).$$

Note that $\mathfrak{n}^+ \oplus \hat{\mathfrak{h}} \oplus \mathfrak{n}^- = \mathcal{L}_{\text{pol}}(\mathfrak{g})$, which is a dense subspace of $\mathcal{L}(\mathfrak{g})$ relative to the Fréchet topology. Moreover, the Lie algebra of \mathcal{U}^\pm is the closure of \mathfrak{n}^\pm in $\mathcal{L}(\mathfrak{g})$ relative to the aforementioned topology. Recall that \mathfrak{n}^+ is generated as a Lie algebra by e_i , $0 \leq i \leq r$. Pick some complex numbers $\chi_0^+, \dots, \chi_r^+$ and consider the Lie algebra homomorphism $\chi^+ : \mathfrak{n}^+ \rightarrow \mathbb{C}$ determined by $\chi^+(e_i) = \chi_i^+$, for all $0 \leq i \leq r$. Since \mathcal{U}^+ is simply connected and its Lie algebra contains \mathfrak{n}^+ as a dense subspace, there is a unique lift $\chi^+ : \mathcal{U}^+ \rightarrow \mathbb{C}^*$ (which is a group homomorphism). In the same way, by picking $\chi_0^-, \dots, \chi_r^- \in \mathbb{C}$, one attaches to them a Lie algebra homomorphism $\chi^- : \mathfrak{n}^- \rightarrow \mathbb{C}$ such that $\chi^-(f_i) = \chi_i^-$, $0 \leq i \leq r$, and then a Lie group homomorphism $\chi^- : \mathcal{U}^- \rightarrow \mathbb{C}^*$. A smooth function $\phi : \mathcal{U}^+ \mathcal{T} \mathcal{U}^- \rightarrow \mathbb{C}$ is called a *Whittaker function* if:

$$(A.2) \quad \phi(u_+ g u_-) = \chi^+(u_+) \phi(g) \chi^-(u_-), \text{ for all } u_\pm \in \mathcal{U}^\pm \text{ and } g \in \mathcal{T},$$

$$(A.3) \quad c.\phi = -h^\vee \phi,$$

where h^\vee is the dual Coxeter number of \mathfrak{g} . Such a function ϕ is clearly determined by its restriction to \mathcal{T} . In turn, for any $g \in T$ and any $t_0 \in \mathbb{R}$,

$$\left. \frac{d}{dt} \right|_{t=t_0} \phi(\exp(tc)g) = -h^\vee \phi(\exp(t_0 c)g).$$

The initial value problem formed by this equation along with the condition $\phi(\exp(tc)g)|_{t=0} = \phi(g)$ has a unique solution. Thus, ϕ is uniquely determined by its values on T , hence the space of Whittaker functions can be naturally identified with $C^\infty(T, \mathbb{C})$.

Equation (A.2) implies readily that for any Whittaker function ϕ and any $0 \leq i, j \leq r$ one has:

$$(A.4) \quad e_i \cdot \phi = \chi_i^+ \phi \text{ on } \mathcal{T}$$

$$(A.5) \quad f_i \cdot \phi = \chi_i^- e^{\alpha_i} \phi \text{ on } \mathcal{T}$$

$$(A.6) \quad f_i \cdot e_i \cdot \phi = \chi_i^- \chi_i^+ e^{\alpha_i} \phi \text{ on } \mathcal{T}$$

$$(A.7) \quad f_i \cdot f_j \cdot \phi = \chi_i^- \chi_j^- e^{\alpha_i} e^{\alpha_j} \phi \text{ on } \mathcal{T}$$

$$(A.8) \quad \text{if } \alpha \text{ is an affine positive nonsimple root and } \xi \in \hat{\mathfrak{g}}_\alpha \text{ then } \xi \cdot \phi = 0 \text{ on } \mathcal{T}.$$

There exists a certain completion $\tilde{U}(\hat{\mathfrak{g}})$ of $U(\hat{\mathfrak{g}})$ such that the center of $\tilde{U}(\hat{\mathfrak{g}})/(c + h^\vee)$ is rich, see [4, 5]. First of all, it contains a degree two Casimir element C , which can be expressed as:

$$(A.9) \quad C = \sum_{i=1}^r \xi_i^2 + 2h_\rho + 2 \sum_{\alpha \in \hat{R}^+} \left(\sum_k f_\alpha^k e_\alpha^k \right).$$

Here ξ_1, \dots, ξ_r is an orthonormal basis of \mathfrak{h} , and for each positive affine root $\alpha \in \hat{R}^+$, the vectors e_α^k are a basis of $\hat{\mathfrak{g}}_\alpha$ and f_α^k a basis of $\hat{\mathfrak{g}}_{-\alpha}$ such that $\langle e_\alpha^k, f_\alpha^\ell \rangle = \delta_{k\ell}$. For $\alpha = \alpha_i$ both $\hat{\mathfrak{g}}_\alpha$ and $\hat{\mathfrak{g}}_{-\alpha}$ are one dimensional and we take $e_{\alpha_i}^1 = e_i$, $f_{\alpha_i}^1 = f_i$, $0 \leq i \leq r$. Finally, ρ denotes the half-sum of all positive roots of $(\mathfrak{g}, \mathfrak{h})$ relative to the basis $\alpha_1, \dots, \alpha_r$ and h_ρ the element of \mathfrak{h} that corresponds to it via the Killing form. Note that even though the second sum in (A.9) is infinite, C belongs to the completion $\tilde{U}(\hat{\mathfrak{g}})$.

To any $\xi \in \mathfrak{t}$ we attach the directional derivative ∂_ξ on $C^\infty(T, \mathbb{C})$ given by

$$(\partial_\xi f)(g) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(t\xi)g)$$

for all $f \in C^\infty(T, \mathbb{C})$ and all $g \in T$. Equations (A.4)-(A.9) above imply that for any Whittaker function ϕ we have $(C \cdot \phi)|_T = D(\phi|_T)$, where D is the differential operator on $C^\infty(T, \mathbb{C})$ given by

$$D = \sum_{i=1}^r \partial_{\xi_i}^2 + 2\partial_{h_\rho} + 2 \sum_{i=0}^r \chi_i^+ \chi_i^- e^{\alpha_i}.$$

Here $\partial_{\xi_1}, \dots, \partial_{\xi_r}, \partial_{h_\rho}$ are derivatives and e^{α_i} is the function $T \rightarrow \mathbb{C}$ given by $\exp(h) \mapsto e^{\alpha_i(h)}$, for all $h \in \mathfrak{h}$, $0 \leq i \leq r$. Observe now that $h_\rho = \sum_{i=1}^r \rho(\xi_i) \xi_i$, hence the composition of D

with $e^{-\rho}$ is

$$\begin{aligned} De^{-\rho} &= \left(\sum_{i=1}^r \partial_{\xi_i}^2 + 2\partial_{h_\rho} + 2 \sum_{i=0}^r \chi_i^+ \chi_i^- e^{\alpha_i} \right) e^{-\rho} \\ &= \sum_{i=1}^r e^{-\rho} (\rho(\xi_i)^2 - 2\rho(\xi_i)\partial_{\xi_i} + \partial_{\xi_i}^2) + 2 \sum_{i=1}^r \rho(\xi_i) e^{-\rho} (-\rho(\xi_i) + \partial_{\xi_i}) + 2 \sum_{i=0}^r \chi_i^+ \chi_i^- e^{\alpha_i} e^{-\rho} \\ &= e^{-\rho} \left(\sum_{i=1}^r \partial_{\xi_i}^2 - \sum_{i=1}^r \rho(\xi_i)^2 + 2 \sum_{i=0}^r \chi_i^+ \chi_i^- e^{\alpha_i} \right). \end{aligned}$$

Since $\sum_{i=1}^r \rho(\xi_i)^2 = \langle \rho, \rho \rangle$ and $\alpha_0 = -\theta$ on \mathfrak{h} , one obtains

$$e^\rho De^{-\rho} = \sum_{i=1}^r \partial_{\xi_i}^2 + 2\chi_0^+ \chi_0^- e^{-\theta} + 2 \sum_{i=1}^r \chi_i^+ \chi_i^- e^{\alpha_i} - \langle \rho, \rho \rangle.$$

Choose χ_i^+ and χ_i^- such that $\chi_i^+ \chi_i^- = -1$ for $i \neq 1$, $\chi_0^+ = -1$, and $\chi_0^- = K$. Then

$$M = \frac{1}{2}(e^\rho De^{-\rho} + \langle \rho, \rho \rangle).$$

The center of $\tilde{U}(\mathfrak{g}'_{\text{aff}})/(c + h^\vee)$ contains $Y_1 := C$ along with Y_2, \dots, Y_r , which are of the form $Y_i = u_i + Y_i^+$, where $u_i \in \text{Sym}(\mathfrak{h})$ are fundamental generators of $\text{Sym}(\mathfrak{h})^W$ and Y_i^+ is a sum of monomials in $U(\mathfrak{g})\mathfrak{n}^+$ of degree at most equal to $\deg u_i$, see [4, 5]. From equations (A.4)-(A.8) one deduces that for any Whittaker function ϕ one has $(Y_i \cdot \phi)|_T = D_i(\phi|_T)$, $1 \leq i \leq r$, where D_i is a differential operator on $C^\infty(T, \mathbb{C})$ which admits a presentation as a polynomial in $Ke^{-\theta}, e^{\alpha_1}, \dots, e^{\alpha_r}, \partial_{\xi_1}, \dots, \partial_{\xi_r}$ (one also uses that $(f_0 \cdot \phi)|_T = \chi_0^- e^{\alpha_0} \phi|_T = Ke^{-\theta} \phi|_T$). Furthermore, the symbol of D_i is $u_i(\partial_{\xi_1}, \dots, \partial_{\xi_r})$. The differential operators $D = D_1, D_2, \dots, D_r$ commute with each other. But then also

$$D'_1 := M = \frac{1}{2}(e^\rho De^{-\rho} + \langle \rho, \rho \rangle), D'_2 := e^\rho D_2 e^{-\rho}, \dots, D'_r := e^\rho D_r e^{-\rho}$$

commute with each other. Each of them is a polynomial in $Ke^{-\theta}, e^{\alpha_1}, \dots, e^{\alpha_r}, \partial_{\xi_1}, \dots, \partial_{\xi_r}$, since $e^\rho \partial_{\xi_i} e^{-\rho} = -\rho(\xi_i) + \partial_{\xi_i}$. Moreover, the symbol of D'_i is $u_i(\partial_{\xi_1}, \dots, \partial_{\xi_r})$ as well.

Let us now consider the “ $ax + b$ ” algebra $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{u}$ corresponding to the Dynkin diagram of \mathfrak{g} extended with the highest root. Its complexification is $\mathfrak{b}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{u}_{\mathbb{C}}$, where one can identify $\mathfrak{a}_{\mathbb{C}} = \mathfrak{h}$. Then $\mathfrak{u}_{\mathbb{C}}$ has a \mathbb{C} -basis X_0, \dots, X_r such that the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{b}_{\mathbb{C}}$ is identically zero on both \mathfrak{h} and $\mathfrak{u}_{\mathbb{C}}$ and for any $h \in \mathfrak{h}$ one has

$$\begin{aligned} [h, X_i] &= \alpha_i(h)X_i, \quad 1 \leq i \leq r \\ [h, X_0] &= -\theta(h)X_0. \end{aligned}$$

Let $\Omega_1, \dots, \Omega_r$ be the elements of $U(\mathfrak{b}_{\mathbb{C}})$ which are obtained from the differential operators D'_1, \dots, D'_r above by making the following replacements:

$$e^{\alpha_i/2} \mapsto \frac{\sqrt{-1}}{2\sqrt{2}} X_i, \quad \sqrt{K} e^{-\theta/2} \mapsto \frac{\sqrt{-1}}{2\sqrt{2}} X_0, \quad \partial_{H_i} \mapsto \frac{1}{2} H_i; \quad 1 \leq i \leq r.$$

In this way

$$M = \frac{1}{2} \sum_{i,j=1}^r \langle \alpha_i, \alpha_j \rangle \partial_{H_i} \partial_{H_j} - K e^{-\theta} - \sum_{i=1}^r e^{\alpha_i}$$

turns into $\frac{1}{8}\Omega$, see equation (1.1). Moreover, if $\theta = m_1\alpha_1 + \dots + m_r\alpha_r$, then for any $1 \leq i, j \leq r$ we have:

$$\begin{aligned} \left[\frac{1}{2} H_i, \frac{\sqrt{-1}}{2\sqrt{2}} X_j \right] &= \delta_{ij} \frac{1}{2} \cdot \frac{\sqrt{-1}}{2\sqrt{2}} X_j, & \left[\frac{1}{2} H_i, \frac{\sqrt{-1}}{2\sqrt{2}} X_0 \right] &= -m_i \cdot \frac{1}{2} \cdot \frac{\sqrt{-1}}{2\sqrt{2}} X_0 \\ [\partial_{H_i}, e^{\alpha_j/2}] &= \delta_{ij} \frac{1}{2} e^{\alpha_j/2}, & [\partial_{H_i}, e^{-\theta/2}] &= -m_i \cdot \frac{1}{2} e^{-\theta/2}. \end{aligned}$$

Thus Ω_i is in $U(\mathfrak{b}_{\mathbb{C}})_{\text{ev}}$ and satisfies the three conditions in Theorem 1.1.

Proposition A.1. *The conclusion of Theorem 1.1 is true in the case when Π is an arbitrary irreducible root system and β is the highest root.*

Proof. The operators $\Omega_1, \dots, \Omega_r$ satisfy the conditions in Theorem 1.1. However, we only know that $\Omega_2, \dots, \Omega_r$ are in $U(\mathfrak{b}_{\mathbb{C}})$, although not necessarily in $U(\mathfrak{b})$. For $i \in \{2, \dots, r\}$ one considers the expansion of Ω_i relative to the PBW basis $X^I H^J$. By writing each coefficient as $a + \sqrt{-1}b$, with $a, b \in \mathbb{R}$, one obtains $\Omega_i = \Omega'_i + \sqrt{-1}\Omega''_i$, where both Ω'_i and Ω''_i are in $U(\mathfrak{b})$. But then $[\Omega'_i + \sqrt{-1}\Omega''_i, \Omega] = [\Omega'_i, \Omega] + \sqrt{-1}[\Omega''_i, \Omega]$, which is equal to 0. This implies that $[\Omega'_i, \Omega] = 0$. Thus $\Omega'_2, \dots, \Omega'_r$ are in $U(\mathfrak{b})$ and satisfy the three conditions in Theorem 1.1. It remains to justify the last statement in the theorem, that is, that $[\Omega'_i, \Omega'_j] = 0$, for all $2 \leq i, j \leq r$. This can be proved as follows. First note that, since both Ω'_i and Ω'_j are in $U(\mathfrak{b})_{\text{ev}}$, their bracket $[\Omega'_i, \Omega'_j]$ is in $U(\mathfrak{b})_{\text{ev}}$ as well. Also, since both Ω'_i and Ω'_j commute with Ω , their bracket $[\Omega'_i, \Omega'_j]$ commutes with Ω as well. Since $\mu([\Omega'_i, \Omega'_j]) = [\mu(\Omega'_i), \mu(\Omega'_j)] = 0$, one can use Proposition 2.1 to deduce that $[\Omega'_i, \Omega'_j]$ is a multiple of $(X_0 X_1^{m_1} \dots X_r^{m_r})^2$. On the other hand the degree of $[\Omega'_i, \Omega'_j]$ is at most equal to $\deg \Omega'_i + \deg \Omega'_j - 1$, which is the same as $\deg u_i + \deg u_j - 1$. The table in Section 2 shows that $\deg u_i + \deg u_j - 1$ is strictly smaller than $2(1 + m_1 + \dots + m_r)$, for all $1 \leq i < j \leq r$ (recall that m_1, \dots, m_r are the coefficients of the highest root expansion). We conclude that $[\Omega'_i, \Omega'_j] = 0$, as required. \square

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